

A Theory of Scattering Based on Free Fields

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Abstract

So far only quasifree fields have been shown to satisfy the Haag-Araki axioms for local algebras of observables; we show from a model in $1 + 1$ dimensions that there can be representations in which two incoming free particles produce a pair of out-going solitons with a positive probability, which can be computed. This happens when the experiment is designed to observe this outcome. It is proposed that the same idea will work in four dimensions.

Contents

1. Introduction to Haag fields
2. Reduction to Abelian Multipliers
3. A Model in $1 + 1$ dimensions
4. Some Remarks in Four Space-time Dimensions

1 Introduction to Haag Fields

It has been extremely difficult to construct solutions to renormalisable quantum field theories that satisfy the Wightman axioms, in four space-time dimensions, except free fields and generalised free fields. It has been conjectured that quantum electrodynamics does not exist; only theories containing non-abelian gauge fields, it is claimed, could exist and give a non-trivial S-matrix. Similar remarks apply to the C^* -algebraic systems of Haag and Araki.

The relation between the Wightman axioms and the C^* -algebras is not clear for a general Wightman theory, but for any free boson field a key result due to Slawny [14] suggests a natural way to construct a set of local C^* -algebras which obey the Haag-Kastler axioms. Consider for example a free scalar quantised field of mass $m > 0$. In any Lorentz frame, the free quantised field ϕ and its time derivative π at constant time (say, time zero) can be smeared in the space variable with a continuous function of compact support, to get self-adjoint operators on Fock space. Thus

$$\phi(g) := \int \phi(0, \mathbf{x}) g(\mathbf{x}) d^3x \quad (1)$$

$$\pi(f) := \int \dot{\phi}(0, \mathbf{x}) f(\mathbf{x}) d^3x \quad (2)$$

have well-defined exponentials, as do their sums; let \mathcal{H} be the space of real solutions $\varphi(t, \mathbf{x})$ to the wave equation with initial values $\varphi(0, \mathbf{x}) = f(\mathbf{x})$ and $\dot{\varphi}(0, \mathbf{x}) = g(\mathbf{x})$. This is a dense subspace of the one-particle space, a complex Hilbert space. The imaginary part of the scalar product, the symplectic structure of the classical field theory, is the Wronskian B of the two solutions, the Lorentz invariant anti-symmetric bilinear expression

$$B(\varphi_1, \varphi_2) := \int d^3x [f_1(\mathbf{x}) g_2(\mathbf{x}) - g_1(\mathbf{x}) f_2(\mathbf{x})]. \quad (3)$$

The expression

$$B(\phi, \varphi) := \phi(g) - \pi(f), \quad (4)$$

the Wronskian between the quantised and the classical solution, is then self-adjoint. Segal uses the operators

$$W(\varphi) := \exp\{iB(\phi, \varphi)\}, \quad (5)$$

and these obey Segal's form of the Weyl relations for the commutation relations of a free quantised field:

$$W(\varphi_1)W(\varphi_2) = W(\varphi_1 + \varphi_2) \exp\left\{-\frac{i}{2}B(\varphi_1, \varphi_2)\right\}. \quad (6)$$

Eq.(6) gives a product to the vector space defined by symbols $W(\varphi)$ as φ runs over the symplectic space \mathcal{H} , irrespective of the representation by operators W . What Slawny [14] did was to prove that the $*$ -algebra obtained by including this product has a unique C^* -norm; this is a norm on the algebra obeying $\|A^*A\| = \|A\|^2$.

We define the Haag field as follows. Let \mathcal{O} be a bounded closed set in \mathbf{R}^4 , of the form of the intersection of a forward and backward light cone. The cones themselves intersect in a two-dimensional ellipse. Let f and g be continuous functions of the points in the interior of the three-dimensional region spanned by this ellipse, and vanishing on the boundary. Then the local C^* -algebra $\mathcal{A}(\mathcal{O})$ is the completion, in the Slawny norm, of the Segal-Weyl algebra generated by such f and g . The algebra of all observables, \mathcal{A} , is then the completion of the inductive limit of all the local algebras. The algebra defined for an arbitrary connected open region of \mathbf{R}^4 is the completion of the union of all $\mathcal{A}(\mathcal{O})$, \mathcal{O} being a subset of the region.

This field nearly obeys the Haag-Kastler [8] axioms; Haag and Kastler assumed that the Poincaré group acted on \mathcal{A} norm-continuously, which we do not. The free field satisfies one more, the split property of Doplicher and Roberts [5]. We use the notation \mathcal{L} for the Poincaré group, which is the semi-direct product of the group of space-time translations, $x \mapsto x+a$, where a is a real four-vector, and the Lorentz group $x \mapsto \Lambda x$. Thus $L = (a, \Lambda)$ will denote a general element of \mathcal{L} . Then the axioms we use are:

1. There is given an automorphism group τ_L of the Poincaré group; this maps $\mathcal{A}(\mathcal{O})$ onto $\mathcal{A}(L\mathcal{O})$.
2. If two regions \mathcal{A}_1 and \mathcal{A}_2 are space-like separated, then the algebras \mathcal{A}_1 and \mathcal{A}_2 commute.
3. The vacuum representation: there exists a representation R_0 of \mathcal{A} , such that there is a unique vacuum state vector, the Poincaré group is continuously represented by unitary operators, and the spectrum of the energy is bounded below.
4. The split property: if $\mathcal{O}_1^- \subset \mathcal{O}_2$ then there exists a sub-algebra \mathcal{N} of type I such that $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{N} \subset \mathcal{A}(\mathcal{O}_2)$; by type I is meant that the weak closure in the vacuum representation is a von Neumann algebra of type I.

Another possible axiom is Haag duality; this fails to hold in our model in one-plus-one dimensions and we shall not use it.

In their set-up, Haag and Kastler give the following explanation of superselection rules; charged states are not in the state-space containing the vacuum, but are states in some other representation R of the algebra \mathcal{A} , which is not quasi-equivalent to R_0 . We mean the following by quasi-equivalence, which is equivalent to the usual definition for the algebras \mathcal{A} arising in quantum field theory; let \mathcal{A} be such a C^* -algebra. A representation of \mathcal{A} , π_1 on

a Hilbert space \mathcal{H}_1 is said to be *quasi-equivalent* to a representation π_2 on a Hilbert space \mathcal{H}_2 , if there exists an isometry $U : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that

$$U\pi_2(A) = \pi_1(A)U \quad (7)$$

holds for all $A \in \mathcal{A}$. We say that an automorphism σ of \mathcal{A} is *spatial* in a representation π of \mathcal{A} on a Hilbert space \mathcal{H} if there exists a unitary operator U_σ on \mathcal{H} such that

$$\sigma(A) = UAU^{-1} \quad (8)$$

holds for all $A \in \mathcal{A}$. We say that U implements the automorphism in this case. Most automorphisms are not spatial.

Haag and Kastler assume that the Poincaré automorphisms are spatial in R , and that the generator of time evolution also has positive spectrum. R is related to the vacuum representation R_0 by an automorphism σ , $A \mapsto \sigma A$ of \mathcal{A} ; this cannot be a spatial automorphism, since if it were, R and R_0 would be equivalent. Clearly, the representation is given by

$$R_\sigma(A) = R_0(\sigma A), \quad (9)$$

as A runs over \mathcal{A} ; this acts on the Hilbert space containing the vacuum, but is not equivalent to the representation R_0 , since the automorphism σ is not implemented by a unitary operator. We do not expect σ to commute with the space-time translations; thus, the automorphisms of \mathcal{A} , $\tau_a\sigma$, $a \in \mathbf{R}^4$, are not the same as $\sigma\tau_a$ in general. Haag showed that one might reveal the existence of Fermions, carrying a charge, by exploring the representations

$$R_n(A) = R_0(\tau_1\sigma \circ \tau_{a_2}\sigma \dots \tau_{a_n}\sigma A), \quad (10)$$

which would define the n -particle states. A little later, Doplicher and Roberts [5] generalised this idea; to get a representation of \mathcal{A} , one can make do with an *endomorphism* rather than an automorphism; one then gets a reducible representation of the algebra \mathcal{A} by using eq (9). Doplicher and Roberts require that the endomorphism, call it σ , obeys $\sigma(I) = I$, so that the dual action of σ on the states preserves normalisation. Thus Doplicher and Roberts use identity-preserving endomorphisms. The unitaries of the commutant of the representation make up the gauge group. Starting with axioms similar to (1), ..., (4), they [5, 7] find that the gauge group must be a compact Lie group. Now, this holds also for the free field algebra, though Doplicher and Roberts assumed that the given system was not the free field. They are stuck, in that no interacting Wightman theory in four dimensions has yet been constructed.

In this paper, we start with the free field as in [17], and try to find what endomorphisms give rise to new states. We note that it is not obvious that the Lorentz group should be implemented in R even if the space-time translations are; more, the space-time group might acquire non-abelian multipliers. In Sect (2) we show that if every one-parameter space-time translation group with a time-like direction has spectrum that is bounded below, then the four-dimensional translation group is represented by unitaries which have multipliers in the centre of $R(\mathcal{A})''$. This proof uses Borchers's theorem [2] in the form proved in Bratteli and Robinson [1]; it arose from a discussion with G. Morchio. We are then reduced to the suggestion of several authors, that the space-time group might be represented with multipliers in the centre.

In Sect. (3) we study the case of a free massless field in $1 + 1$ dimensions, following [18]. This model has been further developed by Ciolli [3]. We show that a soliton pair of states with opposite charges does lie in Fock space, and converges $*$ -weakly to an out-going pair in a new representation. The pair is created from a state in Fock space by the very act of asking the question, is a pair present at $t = \infty$?

In Sect.(4) we suggest a programme that might lead to similar results in four space-time dimensions.

2 Reduction to Abelian Multipliers

It is usually required that the endomorphism, denoted by σ above, should be such that the Poincaré group be spatial in the representation R_σ . However, with particles of zero mass, it might not be true. In any case, we shall just assume that space-time translations are symmetries in R_σ ; that is, are each given by an isometric operator with transition probabilities that are measurable functions of the group parameters; then Wigner's analysis can be applied. Now, R_σ is reducible if σ is not an automorphism; thus the commutant $R_\sigma(\mathcal{A})'$ of the representation contains non-commuting unitaries, and so possible multipliers of the group \mathbf{R}^4 might be non-abelian [15, 16]. It is well known that a one-parameter group of automorphisms, if spatial in a representation, has only trivial multipliers [9]. It has been suggested that conditions might be such that the multiplier is abelian. Indeed, there does exist a natural condition which ensures this.

Theorem 11 *Let \mathcal{A} be a C^* -algebra and τ_a be a group action of \mathbf{R}^4 by automorphisms. Let $A \mapsto R(A)$ be a representation of \mathcal{A} such that the group action is weakly measurable. Suppose that for each time-like one-parameter subgroup of \mathbf{R}^4 , the automorphisms are implemented (in the representation*

R) by a continuous one-parameter unitary group, whose self-adjoint generator is bounded below. Then the group \mathbf{R}^4 is projectively represented by unitary operators with abelian multipliers.

Proof. Borchers's theorem [2] was modified by Bratteli and Robinson [1] to the form: let \mathcal{A} be a C^* -algebra on a separable Hilbert space, τ_t a one-parameter group of automorphisms of \mathcal{A} , implemented by the continuous one-parameter unitary group $t \mapsto U(t)$. Then there exists a continuous unitary group $t \mapsto V(t)$ in the weak closure of \mathcal{A} which implements τ_t .

We apply this result to four independent one-parameter timelike one-parameter groups of space-time translations. Choose four linearly independent time-like vectors a_i , $i = 1, \dots, 4$. The generators are bounded below, and so can be replaced by unitary operators in the weak closure. The multipliers, which are expressed as

$$\omega(a_i, a_j) = U(a_i)U(a_j)U(a_i + a_j)^{-1}, \quad (12)$$

shows that for each pair of our four time-like vectors we have $\omega(a_i, a_j)$ lying in $R(\mathcal{A})''$; but these multipliers also lie in $R(\mathcal{A})'$, so must lie in the centre. For any $\lambda \in \mathbf{R}$ we may implement $x \mapsto x + \lambda a_i$ by $U(\lambda a_i) := U(a_i)^\lambda$, for any measurable choice of the branch. Since $R(\mathcal{A})''$ is a von Neumann algebra, we have that $U(\lambda a_i) \in R(\mathcal{A})''$. Now, these group elements generate the group \mathbf{R}^4 , and for any translation y we have unique λ_i , $i = 1, \dots, 4$ such that $y = \lambda_j a_j$; we may define $U(y) := U(\lambda_1 a_1) \dots U(\lambda_4 a_4)$ which implements the automorphism y and lies in $R(\mathcal{A})''$. We prove the theorem by using eq.(12) for any two elements of \mathbf{R}^4 , which shows that $\omega(y_1, y_2)$ lies in the centre.

3 A Model in One-Plus-One Dimensions

The existence of Wightman theories with interaction in $1 + 1$ -dimensions [6] means that it has not been necessary to consider our idea in this case; however, in view of the difficulty, if not the impossibility, of there existing a Wightman theory in four space-time dimensions, it is worth while pointing out the following model.

Consider the Wightman theory of a scalar massless free field $\phi(x, t)$ in $1 + 1$ dimensions. This does not exist as a Wightman theory, but the system given by its space-time derivatives, $\phi_\mu := \partial_\mu \phi$, does. We take this derivative, $\mu = 0, 1$, to define the observable Wightman fields. The smeared fields ϕ_μ at time zero, obey a form of the CCR which can be written in Segal form. We [18] get a Haag field, and show that it obeys axioms 1, 2 and 3. We

consider new representations of the form

$$\partial_x \phi_\sigma = \partial_x \phi + \partial_x \varphi \quad (13)$$

$$\partial_t \phi_\sigma = \partial_t \phi + \varpi. \quad (14)$$

Here, φ and ϖ are real-valued smooth functions, and such that $\partial_x \varphi$ and ϖ have compact support. It is known that the representation obtained this way is equivalent to the Fock representation if and only if the classical solution determined by the initial values φ, ϖ lies in the one-particle space. We showed [18] that there exists a two-parameter family of superselection rules, labelled by “charges” Q, Q' say; these can be any pair of real numbers. If they are both zero, then the automorphism is spatial in the free Fock representation. The set of φ allowed consists of functions such that $\partial_x \varphi \in \mathcal{D}$, and the set of ϖ is \mathcal{D} itself, Schwartz space; this can lead to states not in Fock space. Two representations with different values of either Q or Q^* are inequivalent; it is thus reasonable to put the discrete topology on the set \mathbf{R}^2 . The dual of this topological space is thus the compact gauge group $U(1) \times U(1)$.

Consider, for example, the choice of $Q = 1, Q' = 1$. A general solution to the wave equation can be written as the sum of a left-going and a right-going wave:

$$f(x, t) = f_L(x + t) + f_R(x - t). \quad (15)$$

We see that a left-going wave can have $Q = 1$ and $Q' = 1$ if $f_R = 0$ and $f_L = \varphi$, $\varpi = \partial_x \varphi$, where $\varphi(x) = 1$ if x is sufficiently large, and $\varphi(x) = 0$ for x sufficiently negative. It follows that there is a state in Fock space, with φ consisting of a right-moving positive bump to the right of space, with $Q = -1$ and $Q' = -1$, and a left-moving negative bump to the left of space, with $Q = 1$ and $Q' = 1$. Let $F(x, t)$ be classical solution with these properties. Then the automorphism is implemented by the unitary operator $W(F) = \exp\{i(\phi(\dot{F}) - \pi(F))\}$.

As time goes by, these solitons move as out-going free particles. There is a non-zero probability P that a given two-particle state $|2\rangle$ in Fock space will lead to this configuration:

$$P = |\langle 2 | W \Psi_0 \rangle|^2 > 0. \quad (16)$$

It is clear that if we look for the free particles, we will see them; no new particles are produced. The charged particles are produced by the setting-up of the procedure to see them.

Further work on this model was done by Ciolli [3]. He proved using Roberts's net cohomology [13] that all possible superselection rules were found in [18].

4 An Attempt in Three + One Dimensions

The electromagnetic field obeys the Maxwell equations

$$\operatorname{div} \mathbf{E} = \rho \quad (17)$$

$$\operatorname{div} \mathbf{B} = 0 \quad (18)$$

$$\partial_t \mathbf{B} = -\operatorname{curl} \mathbf{E} \quad (19)$$

$$\partial_t \mathbf{E} = \operatorname{curl} \mathbf{B} + \mathbf{j} \quad (20)$$

The free-field arises when ρ and \mathbf{j} vanish; the classical electromagnetic wave is described by a transverse free \mathbf{E}, \mathbf{B} . That is, \mathbf{E} and \mathbf{B} are both orthogonal to the momentum of the wave. There are two states, labelled by the polarisation, for each momentum. The set of such solutions form a real Hilbert space, with a symplectic form and a complex structure. The action of the Poincaré group is unitary, the representation being of mass zero and helicity ± 1 . The three components of $\operatorname{curl} \mathbf{E}$ are transverse, even when ρ is not zero. For, the distribution $\operatorname{curl} \mathbf{E}$ has three components. The x -component is $\partial_z E_y - \partial_y E_z$; thus, $\operatorname{curl} \mathbf{E}$, smeared with the three-vector \mathbf{f} , is the space of operators

$$\operatorname{curl} \mathbf{E}(\mathbf{f}) = \mathbf{E} \cdot \operatorname{curl} \mathbf{f}$$

whereas the longitudinal part of the field is of the form $\mathbf{E} \cdot \nabla g$. Since the set $\operatorname{curl} \mathbf{f}$ is disjoint from the set of ∇g except for 0, we have shown that $\operatorname{curl} \mathbf{E}$ is transverse.

Smeared with test functions \mathbf{f} in $\mathcal{D}(\mathbf{R}^3)$, the functions $\operatorname{curl} \mathbf{f}$ are dense in the one-particle space. We define the local C^* -algebra $\mathcal{A}(\mathcal{O})$ using Slawny's theorem, using test-functions in $\mathcal{D}(\mathcal{O})$. The global C^* -algebra \mathcal{A} is the completion of the union of all such algebras for bounded regions in space-time. Let R be the relativistic Fock representation of the transverse electromagnetic field.

We seek an identity-preserving endomorphism σ of \mathcal{A} so that the representation obtained by $R_\sigma(A) = R(\sigma(A))$ is disjoint from the representation R_0 . More, we need that the space-time automorphisms of \mathcal{A} should be spatial in R_σ , and that any one-dimensional time-like translation group should be continuous, and that its generator should be bounded below. The dynamics of the operators in R_σ is given by the free automorphism group of the

free field. However it is not a trivial dynamics, so we hope. The Hamiltonian is not a bounded operator, and neither are the field operators. So these are not in the C^* -algebra, and their algebraic properties might not be preserved if we change to an inequivalent representation. The commutator of these gives the time evolution of the field operator. However, the Lie algebra of such commutators might not be preserved under the endomorphism: there might be new terms, an induced interaction. This is due to the anomalies that arise in commutators. Another possibility, which changes the equations of motion, is to change coordinates of space-time by a smooth but non-linear map. This might lead to a new representation, but it is not clear that the space-time translations would be spatial in the new representation.

Leyland and Roberts [10] have used the theory of sheaf cohomology to study the possible two-cocycles of some free classical fields in Minkowski space. They conclude that for the scalar Klein-Gordon real field, the two-cohomology group is trivial, while for the free Maxwell field there is a two-parameter family of two-cocycles, labelled by electric and the magnetic charge. They also showed that the classical four-potential, A_μ , obeying the subsidiary condition $\partial^\mu A_\mu = 0$ and the wave equation $(\partial_0^2 - \Delta)A_\mu = 0$, showed a one-parameter family of electric charges. It is not clear from their remarks that this holds in the quantum case, which requires non-commuting operators for the fields; however, it does hold. As we did in $1 + 1$ dimensions, we can add this classical solution to the free quantised field, to generate an automorphism of the free field algebra. When we add a cocycle which is not a coboundary, we get a new representation. Leyland and Roberts do not consider the condition that the Maxwell field should be transverse, nor the requirement that the new representations found should have energy bounded below. The latter condition can be satisfied if we require that the solution should extend to the point at infinity, as in the methods described by Ward and Wells [21]. This is possible only for a subset of the solutions, namely, those with integer charge. Thus, the problem with continuous charge can be solved in this way. We can remove the occurrence of magnetic charge by requiring the existence of a potential A_μ . However, this work leads to sectors with zero mass, since there is no mass-parameter in the model. This leads to doubts that it is an electron.

Of interest is the model of Prasad and Sommerfield [12]. They explicitly construct a smooth solution of a free massive boson field in a non-abelian gauge field, and the electromagnetic part of the gauge field has a magnetic pole as well as an electric pole. The energy of the solution is finite. The rigorous treatment [21] concerns the analytic continuation from Minkowski to Euclidean space \mathbf{R}^4 . It mostly assumes that the Euclidean gauge field

is dual or anti-dual $\mathbf{E} = \pm i\mathbf{H}$, though the book also deals with some non-self-dual electromagnetic fields. Donaldson [4] has pointed out that in four dimensions, in the Euclidean formulation, and in the case of self-dual electromagnetic tensors, the second sheaf cohomology group is non-trivial. He remarks that this would furnish \mathbf{R}^4 with new differential structures. From the point of view of the second quantised theory, the C^* -algebra of the electromagnetic field reveals the charge in its equations of motion in the corresponding representation.

The book [21] deals with the classical version of this problem. However, for linear fields, this is close to the quantum version, as we saw in [18]; we use the classical solution to get the displaced Fock representations. Further, the non-linearity of the gauge field in classical field theory can sometimes be linearised by a suitable change of coordinates. The representations obtained by smooth invertible change of coordinates are generally spatial in Fock space; they would produce unstable particles instead of superselected states. The Euclidean approach of Symanzik [19] and Nelson [11] might be the way to proceed; a coordinate change in Euclidean variables could lead to the correct version of the relation between the Fock and non-Fock representations.

In $4 + 1$ dimensions, Vasiliev has shown that a four-dimensional change of coordinates leads us to the soliton, which obeys a Dirac equation.

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